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STATIC AND DYNAMIC ANALYSES OF ANISOTROPIC PLATES WITH CORNER POINTS

R. O. Grossi

Applied Mathematics Program, Research Member of CONICET, School of Engineering, Universidad Nacional de Salta Avenida Bolivia 5150, 4400 *Salta, República Argentina. E-mail: grossiro@unsa.edu.ar*

AND

L. Lebedev

Department of Mechanics and Mathematics, Institute of Mechanics and Applied Mathematics, Rostov State University, Ul. Zorge 5, 344090 *Rostov on Don, Russia*

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1. INTRODUCTION

Differential equations and boundary conditions for physical phenomena are often obtained from physical principles by means of variational techniques. The necessary conditions for the existence of extrema of a functional lead to the natural boundary conditions and to the Euler differential equation, which involve derivatives of an order higher than the order of the derivatives appearing in the functional. Substantial literature has been devoted to the formulation, by means of variational techniques, of boundary value and eigenvalue problems in the statics and dynamics of isotropic plates [1–6]. The natural boundary conditions of certain structural systems are not easily formulated without the use of the calculus of variations. The equations of an isotropic plate with given shear forces and bending moments along the boundary can be found in any textbook on the theory of plates and shells [7, 8]. Commonly, the formulation employs local co-ordinates related to the boundary curve. One of the axes of the frame is the vector $\bar{n}$, corresponding to the unit exterior normal. If the boundary has an angle α, the vector $\bar{n}$ rotates through this angle and so is not continuous. It seems that no published paper or book considers this special case for anisotropic plates, giving the general impression that it is unimportant how we choose to appoint conditions at a corner point of the boundary. In this paper we shall obtain the natural conditions which correspond to an anisotropic plate with a corner point under various boundary conditions.

On the other hand, Hamilton's principle is used to derive the equation of motion and the corresponding boundary conditions for the dynamic case. A particularly important plate problem involves a rectangular plate with a free corner formed by the intersection of two free or simply supported edges. The determination of natural frequencies in the transverse vibration of an isotropic rectangular plate is a problem that has been extensively studied by several researchers. Leissa's works [9, 10] constitute excellent compilations of the pertinent literature. There is comparatively limited information on the vibration of anisotropic plates. The present paper also deals with the application of the Ritz method to the determination of the natural frequencies of a rectangular anisotropic plate with a free corner formed by the

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intersection of two free edges. The resulting algorithm permits the analysis of anisotropic, orthotropic and isotropic materials. Accurate values can be obtained by incrementing the number of orthogonal polynomials, and the entire algorithm can be implemented on a personal computer. The software constitutes a useful tool in design work because of the great number of vibrating anisotropic plate problems that can be solved.

2. STATEMENT OF THE PROBLEM

In the theory of the bending of thin plates considered by Lekhnitskii in his excellent book [11], the bending moments M_1, M_2, the twisting moment H_{12} and the transverse shear forces N_1, N_2 are given, respectively, by

$$M_1 = -\left(D_{11}\frac{\partial^2 w}{\partial x_1^2} + D_{12}\frac{\partial^2 w}{\partial x_2^2} + 2D_{16}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \quad (1)$$

$$M_2 = -\left(D_{12}\frac{\partial^2 w}{\partial x_1^2} + D_{22}\frac{\partial^2 w}{\partial x_2^2} + 2D_{26}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \quad (2)$$

$$H_{12} = -\left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2} + 2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right), \quad (3)$$

$$N_1 = -\left(D_{11}\frac{\partial^3 w}{\partial x_1^3} + 3D_{16}\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + D_{26}\frac{\partial^3 w}{\partial x_2^3}\right), \quad (4)$$

$$N_2 = -\left(D_{16}\frac{\partial^3 w}{\partial x_1^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x_1^2 \partial x_2} + 3D_{26}\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + D_{22}\frac{\partial^3 w}{\partial x_2^3}\right), \quad (5)$$

where $w = w(x_1, x_2)$ denotes the deflection of the mid-surface of the plate and D_{kl} the rigidities of the anisotropic plate. The energy functional for the plate deformed by a load of density $q = q(x_1, x_2)$ acting on R, an external force of density $p = p(s)$ and a bending moment with density $m = m(s)$ acting on the boundary Γ, is given by

$$E(w) = \frac{1}{2}\iint_R \left[D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 \right.$$
$$\left. + 4\frac{\partial^2 w}{\partial x_1 \partial x_2}\left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2}\right)\right] \mathrm{d}x_1\,\mathrm{d}x_2 - \iint_R q w\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$- \int_\Gamma \left(pw - m\frac{\partial w}{\partial n}\right)\mathrm{d}s. \quad (6)$$

Let the boundary Γ be composed of two parts, $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is rigidly clamped, while Γ_2 is simply supported or free, and contains a corner point P_3 as shown in Figure 1(a) and 1(b).

It is well known that the minimum of the functional (6) on the smooth functions that satisfy the clamping conditions is attained when the deflection w is a solution of the problem

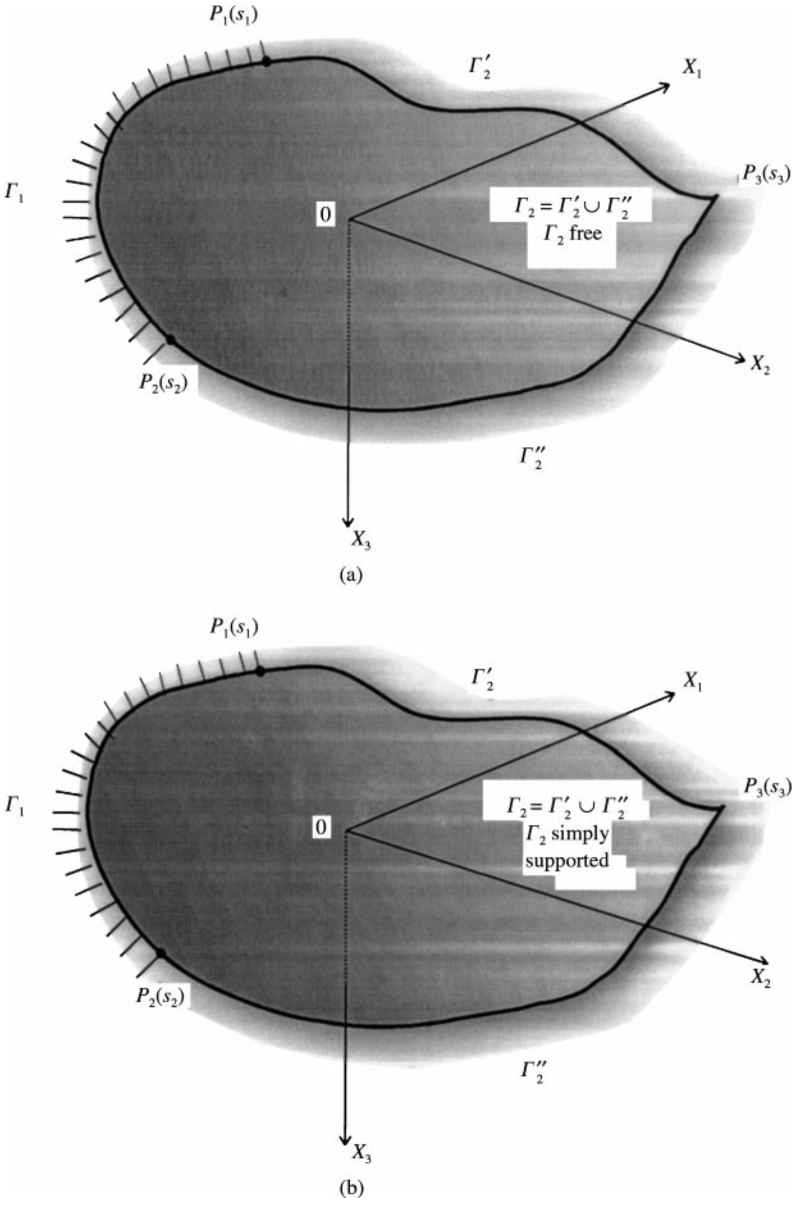

Figure 1. Anisotropic plate with a corner point in Γ_2 which is (a) free and (b) simply supported.

of equilibrium of the plate. The equation for this problem is known [11]:

$$D_{11}\frac{\partial^4 w}{\partial x_1^4} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4 w}{\partial x_2^4} = q. \quad (7)$$

It is the purpose of this paper to use the calculus of variations to obtain the equation of motion and the natural boundary conditions in Γ, and particularly those at the corner point P_3 of Γ_2 and at the bordering points of Γ_1 and Γ_2.

The differential equation (7) and the corresponding boundary conditions are derived by setting the first variation $\delta E(w)$ of functional (6) equal to zero. Since Γ_1 is rigidly clamped the geometric boundary conditions are given by

$$w(s)|_{\Gamma_1} = 0, \qquad \left.\frac{\partial w(s)}{\partial n}\right|_{\Gamma_1} = 0, \ (s \in \Gamma_1). \tag{8}$$

In consequence, the virtual displacement v must satisfy

$$v(s)|_{\Gamma_1} = 0, \qquad \left.\frac{\partial v(s)}{\partial n}\right|_{\Gamma_1} = 0. \tag{9}$$

Let $w \in C^4(R)$ be a minimizer of the functional (6) and consider $E(w + tv)$ at a fixed virtual displacement $v \in C^2(R)$ as a function of the real parameter t. Since it takes a minimum at $t = 0$ we have

$$\begin{aligned}\delta E(w) = \frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{1}{2}\iint_R \Bigg[& D_{11}\left(\frac{\partial^2}{\partial x_1^2}(w+tv)\right)^2 + D_{22}\left(\frac{\partial^2}{\partial x_2^2}(w+tv)\right)^2 \\ & + 2D_{12}\frac{\partial^2}{\partial x_1^2}(w+tv)\frac{\partial^2}{\partial x_2^2}(w+tv) + 4D_{66}\left(\frac{\partial^2}{\partial x_1 \partial x_2}(w+tv)\right)^2 \\ & + 4\left(\frac{\partial^2}{\partial x_1 \partial x_2}(w+tv)\right)\left(D_{16}\frac{\partial^2}{\partial x_1^2}(w+tv) + D_{26}\frac{\partial^2}{\partial x_2^2}(w+tv)\right)\Bigg]\mathrm{d}x_1\,\mathrm{d}x_2 \\ & - \iint_R q(w+tv)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_\Gamma \left[p(s)(w+tv) - m(s)\frac{\partial}{\partial n}(w+tv)\right]\mathrm{d}s\Bigg\}\Bigg|_{t=0} = 0. \end{aligned} \tag{10}$$

Now we invoke Green's formula,

$$\iint_R u\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_\Gamma uvn_i\,\mathrm{d}s - \iint_R v\frac{\partial u}{\partial x_i}\,\mathrm{d}x, \ u, v \in C^{(1)}(R),$$

where n_i denotes the components of the normal exterior to the boundary of R; two applications of this to equation (10) give

$$\begin{aligned}\delta E(w) = {} & D_{11}\left\{\iint_R \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)v\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_\Gamma \left(\frac{\partial^2 w}{\partial x_1^2}\right)\left(\frac{\partial v}{\partial x_1}\right)n_1\,\mathrm{d}s - \int_\Gamma \frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2}\right)vn_1\,\mathrm{d}s\right\} \\ & + D_{22}\left\{\iint_R \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)v\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_\Gamma \left(\frac{\partial^2 w}{\partial x_2^2}\right)\left(\frac{\partial v}{\partial x_2}\right)n_2\,\mathrm{d}s - \int_\Gamma \frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)vn_2\,\mathrm{d}s\right\} \\ & + D_{12}\left\{\iint_R \left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_2^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\right]v\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_\Gamma \left(\frac{\partial^2 w}{\partial x_2^2}\right)\left(\frac{\partial v}{\partial x_1}\right)n_1\,\mathrm{d}s\right.\end{aligned}$$

$$+\int_\Gamma\left(\frac{\partial^2 w}{\partial x_1^2}\right)\left(\frac{\partial v}{\partial x_2}\right)n_2\,\mathrm{d}s-\int_\Gamma\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_2^2}\right)n_1+\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)n_2\right]v\,\mathrm{d}s\Bigg\}$$

$$+4D_{66}\Bigg\{\iint_R\left[\frac{1}{2}\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)+\frac{1}{2}\frac{\partial^2 w}{\partial x_2\partial x_1}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\right]v\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$+\frac{1}{2}\int_\Gamma\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\left(\frac{\partial v}{\partial x_2}\right)n_1\,\mathrm{d}s+\frac{1}{2}\int_\Gamma\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\left(\frac{\partial v}{\partial x_1}\right)n_2\,\mathrm{d}s$$

$$-\frac{1}{2}\int_\Gamma\left[\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)n_2+\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)n_1\right]v\,\mathrm{d}s\Bigg\}$$

$$+2D_{16}\Bigg\{\iint_R\left[\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)+\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\right]v\,\mathrm{d}x_1\,\mathrm{d}x_2-\int_\Gamma\left[\left(\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\right.\right.$$

$$\left.\left.+\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\right)n_1+\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1^2}\right)n_2\right]v\,\mathrm{d}s+\int_\Gamma\left(\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)n_1\right.$$

$$\left.+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)n_2\right)\frac{\partial v}{\partial x_1}\,\mathrm{d}s+\int_\Gamma\frac{1}{2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\frac{\partial v}{\partial x_2}\,n_1\,\mathrm{d}s\Bigg\}+2D_{26}\Bigg\{\iint_R\left[\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)\right.$$

$$\left.+\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)\right]v\,\mathrm{d}x_1\,\mathrm{d}x_2-\int_\Gamma\left[\left(\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)+\frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_2^2}\right)\right)n_2\right.$$

$$\left.+\frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)n_1\right]v\,\mathrm{d}s+\int_\Gamma\left(\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)n_2+\frac{1}{2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)n_1\right)\frac{\partial v}{\partial x_2}\,\mathrm{d}s$$

$$+\int_\Gamma\frac{1}{2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)\frac{\partial v}{\partial x_1}\,n_2\,\mathrm{d}s\Bigg\}-\iint_R qv\,\mathrm{d}x_1\,\mathrm{d}x_2-\int_\Gamma\left(pv-m\frac{\partial v}{\partial n}\right)\mathrm{d}s=0. \tag{11}$$

2.1. CASE OF SMOOTH BOUNDARY

First, we consider the case when the plate has no corner points, so we suppose that the boundary Γ is smooth and that Γ_2 is free. In order for the functional (6) to have a minimum, we must require that $\delta E(w)=0$ for all admissible virtual displacements v, and in particular for all admissible v satisfying on the whole contour Γ the conditions

$$v(s)|_\Gamma=0,\qquad \left.\frac{\partial v(s)}{\partial n}\right|_\Gamma=0. \tag{12}$$

For such functions equation (11) reduces to

$$\delta E(w)=\iint_R\Bigg\{D_{11}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)+D_{22}\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)+2D_{12}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)+4D_{66}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_2^2}\right)$$

$$+4D_{16}\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)+4D_{26}\frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2 w}{\partial x_1\partial x_2}\right)-q\Bigg\}v\,\mathrm{d}x_1\,\mathrm{d}x_2=0. \tag{13}$$

Since v is an arbitrary smooth function satisfying conditions (12), the Fundamental Lemma of the Calculus of Variations can be applied and we obtain equation (7).

Next we remove restrictions (12), and since w must satisfy equation (7) equation (11) reduce to

$$\delta E(w) = -\int_{\Gamma_2}\left\{D_{11}\frac{\partial^3 w}{\partial x_1^3} + D_{12}\frac{\partial^3 w}{\partial x_1 \partial x_2^2} + 2D_{16}\left(\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) + \frac{1}{2}\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1^2}\right)\right)\right.$$

$$\left. + 2D_{26}\left(\frac{1}{2}\frac{\partial^3 w}{\partial x_2^3}\right) + 4D_{66}\left(\frac{1}{2}\frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right)\right\} n_1 v\,\mathrm{d}s - \int_{\Gamma_2}\left\{D_{12}\frac{\partial^2 w}{\partial x_1^2 \partial x_2}\right.$$

$$+ D_{22}\frac{\partial^3 w}{\partial x_2^3} + 2D_{16}\left(\frac{1}{2}\frac{\partial^3 w}{\partial x_1^3}\right) + 2D_{26}\left(\frac{\partial}{\partial x_2}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) + \frac{1}{2}\frac{\partial}{\partial x_1}\left(\frac{\partial^2 w}{\partial x_2^2}\right)\right)$$

$$\left. + 4D_{66}\left(\frac{1}{2}\frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right)\right\} n_2 v\,\mathrm{d}s + \int_{\Gamma_2}\left\{\left(D_{11}\frac{\partial^2 w}{\partial x_1^2} + D_{12}\frac{\partial^2 w}{\partial x_2^2} + 2D_{16}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)\frac{\partial v}{\partial x_1} n_1\right.$$

$$+ \left(D_{22}\frac{\partial^2 w}{\partial x_2^2} + D_{12}\frac{\partial^2 w}{\partial x_1^2} + 2D_{26}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)\frac{\partial v}{\partial x_2} n_2 + \left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2}\right.$$

$$\left. + 2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)\left(\frac{\partial v}{\partial x_2}\right) n_1 + \left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2} + 2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)\left(\frac{\partial v}{\partial x_1}\right) n_2$$

$$\left. - pv + m\frac{\partial v}{\partial n}\right\}\mathrm{d}s = 0. \tag{14}$$

Now if in equation (14) we use the notations of equations (1)–(5) and introduce local co-ordinates (s, n) by means of the equations

$$\frac{\partial v}{\partial x_1} = \frac{\partial v}{\partial n} n_1 - \frac{\partial v}{\partial s} n_2, \qquad \frac{\partial v}{\partial x_2} = \frac{\partial v}{\partial n} n_2 + \frac{\partial v}{\partial s} n_1,$$

we have

$$\int_{\Gamma_2}\{N_1 n_1 + N_2 n_2 - p\} v\,\mathrm{d}s + \int_{\Gamma_2}\{-M_1 n_1^2 - M_2 n_2^2 - 2H_{12} n_1 n_2 + m\}\frac{\partial v}{\partial n}\mathrm{d}s$$

$$+ \int_{\Gamma_2}\{(M_1 - M_2) n_1 n_2 + H_{12}(n_2^2 - n_1^2)\}\frac{\partial v}{\partial s}\mathrm{d}s = 0. \tag{15}$$

Since Γ is smooth, integration by parts with respect to s in the last integral yields

$$\int_{\Gamma} F\frac{\partial v}{\partial s}\mathrm{d}s = \int_o^l F\frac{\partial v}{\partial s}\mathrm{d}s = Fv|_o^l - \int_o^l \frac{\partial F}{\partial s} v\,\mathrm{d}s = -\int_o^l \frac{\partial F}{\partial s} v\,\mathrm{d}s, \tag{16}$$

$$\text{where } F = (M_1 - M_2) n_1 n_2 + H_{12}(n_2^2 - n_1^2).$$

Substitution into equation (15) gives

$$\int_{\Gamma_2} \{-M_1 n_1^2 - M_2 n_2^2 - 2H_{12} n_1 n_2 + m\} \frac{\partial v}{\partial n} \,\mathrm{d}s + \int_{\Gamma_2} \left\{N_1 n_1 + N_2 n_2 - p - \frac{\partial F}{\partial s}\right\} v \,\mathrm{d}s = 0.$$

Since we can independently choose v and $\partial v/\partial n$, we get the following natural boundary conditions which establish requirements on the bending moment and on the shear force respectively:

$$m(s) = M_1 n_1^2 + M_2 n_2^2 + 2H_{12} n_1 n_2|_{\Gamma_2}, \qquad s \in \Gamma_2, \tag{17}$$

$$p(s) = N_1 n_1 + N_2 n_2 + \frac{\partial}{\partial s}((M_2 - M_1) n_1 n_2 + H_{12}(n_1^2 - n_2^2))|_{\Gamma_2}, \qquad s \in \Gamma_2. \tag{18}$$

On the other hand, at the bordering points P_1, P_2 the functions v must satisfy

$$v(s_i) = 0, \qquad \frac{\partial v(s_i)}{\partial n} = 0, \quad i = 1, 2. \tag{19}$$

2.2. CASE OF PRESENCE OF A CORNER POINT

Now let us assume that there is a corner point P_3 in the part Γ_2 of the boundary Γ, that this part of the boundary is free as shown in Figure 1(a), and that there is a point force p_0 applied in P_3. In this case the result of integrating by parts equation (16) is not valid. The functions $n_1(s)$, $n_2(s)$ are not continuous, so integrating by parts and taking into account in equation (15) the term which corresponds to the energy $-p_0 w|_{s_3}$, we would get additional terms at the corner point:

$$\int_{\Gamma_2} \{-M_1 n_1^2 - M_2 n_2^2 - 2H_{12} n_1 n_2 + m\} \frac{\partial v}{\partial n} \,\mathrm{d}s + \int_{\Gamma_2} \left\{N_1 n_1 + N_2 n_2 - p - \frac{\partial F}{\partial s}\right\} v \,\mathrm{d}s$$
$$+ Fv|_{s_3 - 0}^{s_3 + 0} - p_0 v|_{s_3} = 0. \tag{20}$$

Now, we can choose the subset of functions v which verify the condition $v(s_3) = 0$, and the Fundamental Lemma leads to the same conditions (17) and (18), which now are valid on Γ_2 except at the point P_3. Because of this, the integral terms in equation (20) are equal to zero, and there remains the additional condition at P_3 which is obtained when we consider functions v which verify $v(s_3) \neq 0$. This condition is given by

$$[(M_1 - M_2) n_1 n_2 + H_{12}(n_2^2 - n_1^2)] v|_{s_3 - 0}^{s_3 + 0} - p_0 v|_{s_3} = 0. \tag{21}$$

Since $v(s_3) \neq 0$, we get the additional condition at point P_3,

$$[(M_1 - M_2) n_1 n_2 + H_{12}(n_2^2 - n_1^2)]|_{s_3 + 0} - [(M_1 - M_2) n_1 n_2$$
$$+ H_{12}(n_2^2 - n_1^2)]|_{s_3 - 0} = p_0. \tag{22}$$

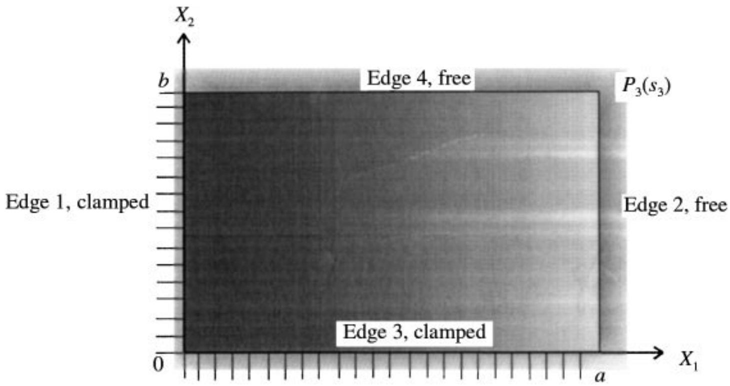

Figure 2. Rectangular anisotropic plate with two free adjacent edges.

Equation (22) is an additional condition at the corner point, which when $p_0 = 0$ demonstrates that the twisting moment is continuous at P_3. When $p_0 \neq 0$ it means that the twisting moment has a jump of value p_0.

For an isotropic rectangular plate, as shown in Figure 2, we have

$$D_{11} = D,\ D_{22} = D,\ D_{12} = \mu D,\ D_{16} = 0,\ D_{26} = 0,\ D_{66} = (D/2)(1 - \mu),$$

where μ is the Poisson ratio, and equation (22) reduces to

$$\left[-D(1-\mu)\frac{\partial^2 w}{\partial x_1 \partial x_2}(n_2^2 - n_1^2)\right]\Bigg|_{s_3+0} - \left[-D(1-\mu)\frac{\partial^2 w}{\partial x_1 \partial x_2}(n_2^2 - n_1^2)\right]\Bigg|_{s_3-0} = p_0.$$

Replacing the values of n_1 and n_2 yields

$$-D(1-\mu)\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\bigg|_{s_3+0} + \frac{\partial^2 w}{\partial x_1 \partial x_2}\bigg|_{s_3-0}\right) = p_0. \tag{23}$$

Equation (23) is a condition at P_3 for the mixed second derivative of w when the force p_0 is not equal to zero.

Now let us assume that Γ_2 is simply supported. In this case the functions v satisfy the condition $v(s)|_{\Gamma_2} = 0$; consequently, from equation (20) we again obtain the natural boundary condition given by equation (17). To sum up, this condition appears to be independent of the existence of the corner point.

3. THE EIGENVALUE PROBLEM

In this section we use Hamilton's principle to derive the equation of motion and the corresponding boundary conditions for an anisotropic plate with a corner point and subjected to an external variable force $q = q(x_1, x_2, t)$. The kinetic energy of the anisotropic plate at time t is given by

$$T(w) = \frac{1}{2}\iint_R \rho h \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x_1\,\mathrm{d}x_2, \tag{24}$$

where h is the plate thickness, ρ the plate density and $w = w(x_1, x_2, t)$. Since the potential energy of deformation of the plate is given by the first integral in equation (6), the corresponding Lagrangian is given by

$$L = T - U = \frac{1}{2}\iint_R \left\{ \rho h \left(\frac{\partial w}{\partial t}\right)^2 - \left[D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 + 4\frac{\partial^2 w}{\partial x_1 \partial x_2}\left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2}\right) - 2qw \right]\right\} \mathrm{d}x_1\,\mathrm{d}x_2. \quad (25)$$

Hamilton's principle requires that on the interval $[t_0, t_1]$, when the positions $w(x_1, x_2, t_0)$ and $w(x_1, x_2, t_1)$ are fixed, the actual motion of the plate makes the action integral $I(w) = \int_{t_0}^{t_1} L\,\mathrm{d}t$ stationary in the space of functions

$D = \{w \in C^4(R \times [t_0, t_1]),\ w(x_1, x_2, t_0)$ and $w(x_1, x_2, t_1)$ prescribed, w satisfy the boundary conditions$\}$,

where R is the domain of the plate and $R \times [t_0, t_1]$ denotes the Cartesian product of R and $[t_0, t_1]$. The variation δI is given by

$$\delta I = \frac{1}{2}\int_{t_0}^{t_1}\iint_R \left\{ -\rho h\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[D_{11}\frac{\partial^4 w}{\partial x_1^4} + D_{22}\frac{\partial^4 w}{\partial x_2^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} - 2q \right]\right\} v\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t + \iint_R \rho h\left(\frac{\partial w}{\partial t}\right) v\Big|_{t_0}^{t_1} \mathrm{d}x_1\,\mathrm{d}x_2 + \int_{t_0}^{t_1}\int_{\Gamma_2} f\,\mathrm{d}s\,\mathrm{d}t. \quad (26)$$

The curvilinear integral in equation (26) is identical to those developed in the statical case since the terms which correspond to the kinetic energy make no contribution to this integral. In consequence, the expression of the function f can be obtained from equation (11).

Since $v(x_1, x_2, t_0) = v(x_1, x_2, t_1) = 0$ as required by Hamilton's principle the second double integral in equation (26) is equal to zero. Now, if we assume

$$v(x_1, x_2, t)|_{\Gamma_2} = 0, \qquad \left.\frac{\partial v(x_1, x_2, t)}{\partial n}\right|_{\Gamma_2} = 0, \quad (27)$$

where t is arbitrary, we obtain

$$\delta I = \frac{1}{2}\int_{t_0}^{t_1}\iint_R \left\{ -\rho h\left(\frac{\partial^2 w}{\partial t^2}\right) - \left[D_{11}\frac{\partial^4 w}{\partial x_1^4} + D_{22}\frac{\partial^4 w}{\partial x_2^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} - 2q \right]\right\} v\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t. \quad (28)$$

Setting equation (28) to zero and using the arbitrariness of the interval $[t_0, t_1]$ and of function v inside $R \times [t_0, t_1]$, we obtain the equation of motion for forced vibrations of the anisotropic plate:

$$D_{11}\frac{\partial^4 w}{\partial x_1^4} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4 w}{\partial x_2^4} + \rho h \frac{\partial^2 w}{\partial t^2} = q. \tag{29}$$

Now removing conditions (27) and noting that w must satisfy equation (29), we obtain from the general expression of δI the same boundary conditions as in the static case, when Γ_2 is free and simply supported respectively.

Let us consider the case of a rectangular anisotropic plate with a free corner formed by the intersection of two free edges and the other two edges clamped as shown in Figure 2. From equation (22) it follows that when the plate executes free or forced vibrations, the additional condition

$$H_{12}|_{s_3+0} + H_{12}|_{s_3-0} = 0 \tag{30}$$

must be satisfied at the corner. Nevertheless, as is well known when dealing with the Ritz method, it is not necessary to subject the co-ordinate functions to the natural boundary conditions [3, 13]. Consequently, since equation (30) constitutes a natural boundary condition, it can be ignored in the construction of the approximation function. The assumed shape function for using the Ritz procedure is given by

$$W(x, y) = \sum_i \sum_j c_{ij} p_i(x) q_j(y), \tag{31}$$

where $p_i(x)$ and $q_j(y)$ are orthogonal polynomials, and c_{ij} are arbitrary coefficients to be determined. The first member of the set, $p_1(x)$, is obtained as the simplest polynomial that satisfies the geometrical boundary conditions. Assume that

$$p_1(x) = \sum_{i=1}^{5} a_i x^{i-1}, \tag{32}$$

where the arbitrary constants a_i are determined by substituting equation (32) into the mentioned boundary conditions. The higher members of the set are obtained by employing the Gram–Schmidt orthogonalization procedure as

$$p_2(x) = (x - B_2)p_1(x), \quad p_k(x) = (x - B_k)p_{k-1}(x) - C_k p_{k-2}(x),$$

where

$$B_k = \frac{\int_0^a x(p_{k-1}(x))^2\,\mathrm{d}x}{\int_0^a (p_{k-1}(x))^2\,\mathrm{d}x}, \qquad C_k = \frac{\int_0^a x p_{k-1}(x) p_{k-2}(x)\,\mathrm{d}x}{\int_0^a (p_{k-2}(x))^2\,\mathrm{d}x}.$$

TABLE 1

Values of the frequency coefficient $\Omega = \omega b^2 \sqrt{\rho h/D_{11}}$ for rectangular anisotropic plate with edges 1 and 3 rigidly clamped and edges 2 and 4 free (see Figure 2). The anisotropy is characterized by the following parameters: $D_{22}/D_{11} = 0{\cdot}115202317$, $D_{12}/D_{11} = 0{\cdot}100812496$, $D_{66}/D_{11} = 0{\cdot}0948810$, $D_{16}/D_{11} = -0{\cdot}24333539$, $D_{26}/D_{11} = -0{\cdot}0120837$

Mode sequence	a/b			
	0·5	1·0	1·5	2·0
1	1·629322221322	3·32815164901	6·95486205824	12·29819205377
2	5·024612579025	9·58943372984	12·59650202035	17·56989276651
3	7·967488728093	19·31923585959	25·19807857486	29·09230327539
4	11·014175047324	22·64339945018	43·42837457421	50·77655691680
5	14·981556360180	25·72370575573	47·43970343944	77·29483739719
6	18·939876323114	36·56770335802	52·17708494851	83·48107988609

The polynomial set along the y direction is also generated using the same procedure. The natural frequencies are obtained from the Rayleigh quotient as

$$\omega^2 = \frac{U_{max}}{T_{max}}. \tag{33}$$

Minimization of the Rayleigh quotient (33) with respect to each parameter c_{ij}, leads to the necessary conditions

$$\frac{\partial}{\partial c_{ij}}(\omega^2) = 0. \tag{34}$$

Substituting the approximate function (31) into equation (34) we obtain

$$\sum_i \sum_j [K_{ijkh} - \Omega^2 M_{ijkh}] c_{ij} = 0, \tag{35}$$

where $\Omega = \omega b^2 \sqrt{\rho h/D_{11}}$ is the non-dimensional frequency parameter.

Table 1 depicts values of the first six natural frequencies of a rectangular anisotropic plate, for different values of the ratio a/b.

4. CONCLUSIONS

The calculus of variations was used to derive the boundary value and eigenvalue problems which describe the static and dynamic behaviours of an anisotropic plate with a corner point in the boundary. Natural boundary conditions at the corner point P_3 and at the bordering points P_1, P_2 have been determined for the cases in which Γ_2 is free and simply supported respectively. The formulation can easily be extended to the case of a plate with various corner points.

Hamilton's principle was used to derive the equation of motion and the corresponding boundary conditions. It has also been determined that when a rectangular anisotropic plate with a free corner formed by the intersection of two free edges executes vibrations, the additional condition $H_{12}|_{s_3+0} + H_{12}|_{s_3-0} = 0$ must be taken into account in the corner.

Finally, natural frequencies of a rectangular anisotropic plate have been studied by using orthogonal polynomials in the Ritz method.

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